Concordance Lecture 4

April 21, 2023 10:46 AM

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Last time: Seifert forms, σ, Δ_t, alg. conc.
This time: alg. conc., Alex modules

1 Algebraic sliceness

Thm Slice knots admit hyperbolic Seifert matrices

$V \sim \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$

3 Corollaries

1. σ conc. inv't & Fox-Milnor
2. $\phi : \mathcal{C} \longrightarrow \mathcal{G}$ is well-defined
3. TOP slice $\Longrightarrow$ ALG slice

Defn A knot K is *algebraically slice* if K admits a hyperbolic Seifert matrix

↳ So if K slice, V is hyperbolic, and K is alg. slice

an equivalent defn uses:

Proposition $[K] \in \ker(\phi)$ iff K admits a hyperbolic Seifert matrix

$\Updownarrow$

V cobordant to hyp. H $\Longleftrightarrow$ $V \oplus -H$ congruent to hyp. H'

V hyperbolic

HW

Lickorish 8.14-8.18

Proof of Thm *will prove smoothly (also holds in TOP)

Let
- K be a knot
- Δ be a slice disk
- F be a Seifert surface
- V be the assoc. Seif. matrix

WTS $V \sim \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$

Lemma 1

$\Delta \cup F$ bounds a cpct, oriented 3-mfld w/ collar $M \times [-1,1]$

(a) Define $f: B^4 \setminus N(\Delta) \longrightarrow S^1$
- do this on knot complement, "captures meridian"
- extend to all of bndry "naturally"
- extend to all of $B^4 \setminus N(\Delta)$ by obstruction thy

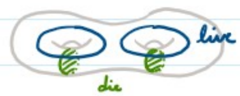
(b) $M = f^{-1}(1)$ is preimage of reg. value

Lemma 2 (½ lives, ½ dies)

If $X = \partial Y^3$ are both cpct and orient'd, then ker and im of $H_1(X;\mathbb{Q}) \xrightarrow{i_*} H_1(Y;\mathbb{Q})$ have dimension $\frac{1}{2}\dim(H_1(X))$

↳ fact you should know about 3-mflds (prove once and forget)

↳ can see w/ handlebodies



So... DRAW PICTURE

extends to above basis for $H_1(\partial M;\mathbb{Q})$

(a) $\exists$ basis $[x_1] \dots [x_{2g}]$ for $H_1(\partial M;\mathbb{Z}) \subseteq H_1(\partial M;\mathbb{Q})$
s.t. $[x_1] \dots [x_g] \in \ker(i_*)$

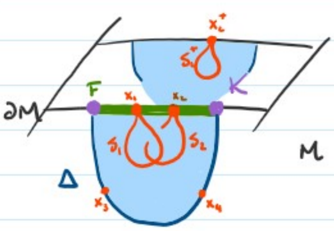
(b) Isotope x_i into F (b/c Δ is a disk!)

(c) $i_*([x_i]) = 0$ in $H_1(M;\mathbb{Q})$ so $\exists n_i \in \mathbb{Z}$
s.t. $n_i[x_i] = 0$ in $H_1(M;\mathbb{Z})$

(d) $n_i x_i$ bounds some surface S_i in $M \subseteq B^4$

(e) $S_i \cap S_j = \varnothing \implies lk(n_i x_i, n_j x_j^+) = 0$

$$\qquad\qquad \| $$
$$n_i n_j\, lk(x_i, x_j^+)$$



$M \subseteq B^4$

$$V \sim \begin{array}{c|cc} & x_1 \dots x_g & \dots \\ \hline \begin{matrix} x_1 \\ \vdots \\ x_g \\ \vdots \end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} 0 & A \\ B & C \end{pmatrix}} \end{array}$$

$\square$

(2) Isomorphism Type of $\mathcal{G}$

Thm $\mathcal{G} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$

(b) Factorisation of $\Delta_t(K)$

(a) Levine-Tristram signature σ_z

$A=\overline{A^T}$

(a) For $z\in\mathbb{C}$ with $|z|=1$,

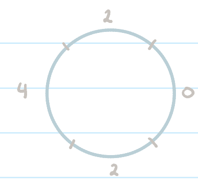
$$(1-z)V + (1-\bar{z})V^T$$

is Hermitian so signature σ_z is in $\mathbb{Z}$

Theorem $\mathcal{C} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\sigma_z} \mathbb{Z}$ is a concordance invariant called Tristram-Levine signature

$A=\overline{A^T}$

FACTS
- Non-singular, hyperbolic, Hermitian $\mathbb{C}$-matrices have $\sigma_z = 0$
  - ↪ Alg slice and $\Delta_z(K)\neq 0 \Rightarrow \sigma_w = 0$
- Recovers classical signature of Murasugi $(\sigma_{-1}=\sigma)$
- Discontinuous at roots of $\Delta_t(K)$
- $\sigma_z(K)=\sigma_{\bar{z}}(K)$



HW Twist knots T_n generate $\mathbb{Z}^\infty$ summand in $\mathcal{G}$.

(b) 2-torsion

Recall $\Delta_t(K)=\Delta_{t^{-1}}(K)$

$$\Rightarrow \Delta_K(t) = \underbrace{(p_1^{r_1}\cdots p_m^{r_m})}_{\text{symmetric}}\overbrace{(q_1^{l_1}\cdots q_n^{l_n})}^{\text{asymmetric}} \qquad p_i(t)=p_i(t^{-1})$$

Defn Let $p\in\mathbb{Z}[t^{\pm 1}]$ be symmetric and $\Delta_K(t)$ factored as above. Define $\phi_p:\mathcal{G}\to\mathbb{Z}_2$ by

$$\phi_p([K]) = r_i \bmod 2 \quad \text{if } p=p_i \text{ for some } i\in\{1,\dots,m\}$$

- Well-defined
- Concordance inv't
- Vanishes on slice knots (by Fox-Milnor)

HW Knots K_n generate $\mathbb{Z}_2^\infty$ summand in $\mathcal{G}$.

similar to the Twist knot

③ Open Problems

- Is $\phi : \mathcal{C} \longrightarrow \mathcal{G}$ split?

  This reduces to finding elements of order 4 in $\mathcal{C}$ mapping to $\mathbb{Z}_4^\infty$

- <u>Conjecture</u> (Gordon) If $[K] \in \mathcal{C}$ has order 2, then $[K]$ contains a negative amphichiral knot $J = -J$

  e.g. $[4_1]$ has order 2 and $4_1 = -4_1$

<u>Lecture 4, part 2</u> Interlude on 3-/4-mfld invts

<u>Intersection forms</u> : X^n compact, oriented, connected

[Assume smooth for convenience

$\exists$ bilinear form
$$\begin{tikzcd} H_p(X,\partial X;\mathbb{Z}) \arrow[rr] \arrow[dr, "\text{P.D.}"'] & & \mathrm{Hom}(H_{n-p}(X;\mathbb{Z}),\mathbb{Z}) \\ & H^{n-p}(X;\mathbb{Z}) \arrow[ur, "\text{Kronecker / ev.}"'] & \end{tikzcd}$$

equivalently, $H_p(X,\partial X)\times H_{n-p}(X)\longrightarrow\mathbb{Z}$

Sometimes also consider
$$\begin{tikzcd} H_p(X) \arrow[rr] \arrow[dr, "i_*"'] & & \mathrm{Hom}(H_{n-p}(X),\mathbb{Z}) \\ & H_p(X,\partial X) \arrow[ur, "K\circ PD"'] & \end{tikzcd}$$
(in LES of pair)

Note: these might be singular e.g. if $\exists$ nontriv. torsion in $H_p(X,\partial X)$

or $\exists$ nontriv $\ker(H_p(X)\to H_p(X,\partial X))$

But $H_p(X,\partial X)/\text{torsion}\times H_{n-p}(X)/\text{torsion}\longrightarrow\mathbb{Z}$

is <u>non-singular</u> i.e. $H_p(X,\partial X)/\text{torsion}\xrightarrow{\cong}\mathrm{Hom}(H_{n-p}(X)/\text{torsion},\mathbb{Z})$

Most interesting case: $p=n-p=\frac{1}{2}n$

Then $Q_X: H_k(X^{2k})\times H_k(X^{2k})\longrightarrow\mathbb{Z}$

is a bilinear form with $Q_X(x,y)=(-1)^kQ_X(y,x)$

i.e. symmetric if k even

skew symm if k odd.

Definition: X^{4k} compact, connected, oriented

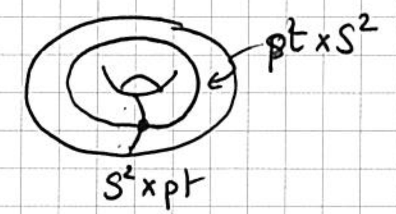
$\sigma(X):=\sigma(Q_X)$

X disconnected, $X=\sqcup X_i$, then $\sigma(X):=\sum\sigma(X_i)$

X_i connected.

Examples: $X = S^4$, $H_2(X) = 0$, Q_X trivial, $\sigma(X) = 0$

$X = S^2 \times S^2$, $H_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by $[S^2 \times pt]$ & $[pt \times S^2]$

Analogy: 

Q_X given by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $\sigma(X) = 0$

$X = \pm \mathbb{C}P^2$, $H_2(X) \cong \mathbb{Z}$ generated by $[\mathbb{C}P^1]$

Q_X given by $[\pm 1]$, $\sigma(X) = \pm 1$.

Fact 1: For $n \leq 4$, every class in $H_i(X^n)$ ~~can~~ be represented by an i-dim submanifold.

Fact 2: Given $a, b \in H_2(X^4)$, rep by (oriented) surfaces $A, B \subseteq X$

$Q_X(a,b) = A \pitchfork B$ signed count.

Fact 3: [Whitehead] Closed $\pi_1 = 1$ 4-mflds are htpy equiv iff isometric intersection forms.

Similarly, $\exists$ <u>linking forms</u>: X^n closed, connected, oriented

$$lk_X : \text{Torsion } H_{n-p}(X^n) \times \text{Torsion } H_{p-1}(X^n) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$$[a] \qquad [b] \longmapsto \frac{1}{m}(b \pitchfork A) \mod \mathbb{Z}$$

$\exists m > 0$ s.t. $m[a] = 0$

so $ma = \partial A$

for some $A \in C_{n-p+1}(X)$

Most interesting case: $n = 2k+1$

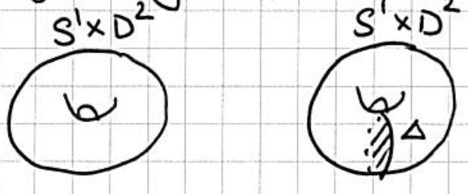
Then $lk_X : \text{Tor } H_k(X) \times \text{Tor } H_k(X) \longrightarrow \mathbb{Q}/\mathbb{Z}$

a bilinear, nonsingular form. Symmetric if k odd

~~an~~ skewsymm if k even

E.g. lens spaces.

By definition, lens spaces are (closed) 3mflds obtained by gluing together two solid tori by a diffeo of the torus

$L(p,q) :=$ 

$(p,q) = 1$

$p\lambda + q\mu \longleftarrow\!\!\mapsto \partial\Delta$

μ = meridian $* \times \partial D^2$

λ = longitude $S^1 \times \underset{\partial D^2}{\underset{\cap}{*}}$

(HW) $H_1(L(p,q)) \cong \mathbb{Z}/p$

$$\ell k_{L(p,q)} : \mathbb{Z}/p \times \mathbb{Z}/p \longrightarrow \mathbb{Q}/\mathbb{Z}$$
$$(a,b) \longmapsto \frac{q}{p} a \cdot b.$$

(Nontrivial) consequence: $L(p,q) \not\hookrightarrow S^4$ unless $L(p,q) \cong S^3, S^1 \times S^2$

[Hantzsche]

[see HW]

[Whitehead] Two lens spaces are htpy equivalent iff isometric linking forms.

$K \subseteq S^3$, $X(K) = S^3 \setminus \mathring{\nu}(K)$ exterior of ~~the~~ knot

Recall $H_*(X(K)) \cong H_*(S^1)$.

We want to extract better invts from $X(K)$.

$\rightsquigarrow$ use covering spaces to access more data in $\pi_1(X(K))$.

Note: $\pi_1(X(K)) \twoheadrightarrow \mathbb{Z}\langle t\rangle$ given by abelianisation

$\mu_K \longmapsto t$ $\quad$ μ_K oriented!

Then get assoc covering space $\widehat{X(K)} \longrightarrow X(K)$ with $\mathbb{Z}\langle t\rangle$ as deck gp.

In other words,

$C_*(\widehat{X(K)})$ has the str. of a $\mathbb{Z}[t^{\pm 1}]$-modul

The homology of this chain complex (as a $\mathbb{Z}[t^{\pm 1}]$-mu

is the Alexander module of K, denoted by $\mathcal{A}(K)$.